I Inverse Function

A. A definition: An inverse function undoes what the function does. The "cube root function" undoes what the "cubing function" does.
Graphically, an inverse function is a reflection of the original across the line $y = x$.
Any restriction on the domain of a function is a restriction on the range of the inverse. This is particularly important when trying to describe the range of a function. You would inspect the inverse function and the restriction on its domain will be the restrictions on the range of the original function.

B. To find the equation of the inverse of a function typically follows the following 3 step process:

Given $f(x) = 2x + 4$

1. Replace the x with z and replace the function name with x.

$$x = 2z + 4$$

2. Solve for z $\quad z = \frac{1}{2}x - 2$

3. Replace z with $f^{-1}(x)$. $\quad f^{-1}(x) = \frac{1}{2}x - 2$

Notice that

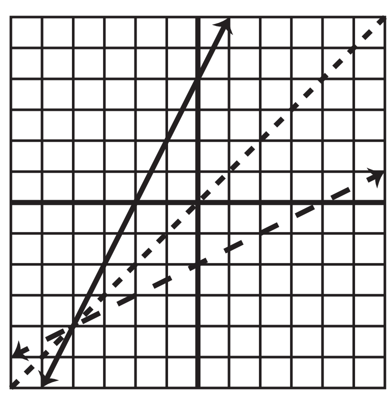
$$f(f^{-1}(x)) = 2\left(\frac{1}{2}x - 2\right) + 4 = x - 4 + 4 = x$$

and

$$f^{-1}(f(x)) = \frac{1}{2}(2x + 4) - 2 = x + 2 - 2 = x$$

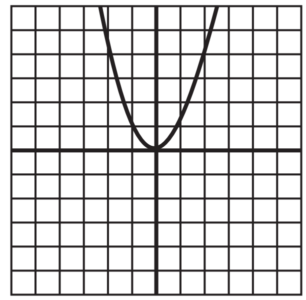
Basically that illustrates that "An inverse function undoes what the function does" and a function undoes what the inverse function does."

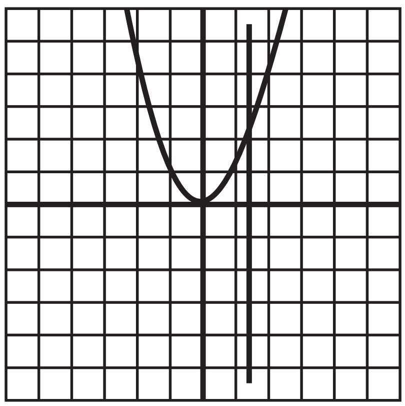
Here the solid line is $f(x) = 2x + 4$. The smallest dashed line is $y = x$ and the third line is $f^{-1}(x) = \frac{1}{2}x - 2$



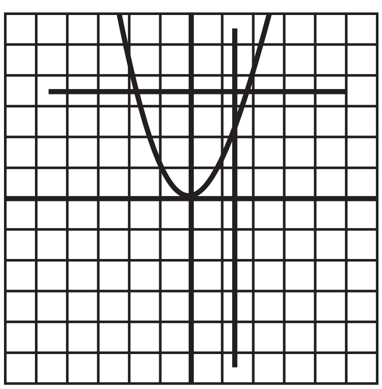
C. Consider the function $y = x^2$:

We know this is a function because for a single x value there is a single y value as is illustrated by the "vertical line test".

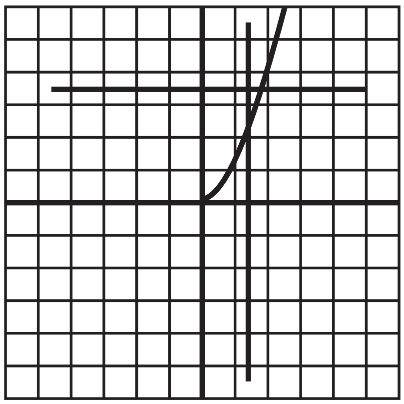




However, we see this function fails the "horizontal line test". That is, for a single value in the range (y-value), there is more than one domain item (x-value). When this happens, the function is said to "not be a 1-to-1 function".



We can *force* our function to be 1-to-1 by restricting the domain of the function. The domain of $y = x^2$ is $D_y \; x \in \mathbb{R}$. We would restrict the domain for our function is to be "all values greater than or equal to zero" $\left(D_f \; x \geq 0 \right)$. Notice the difference in the function and the vertical and horizontal test lines:

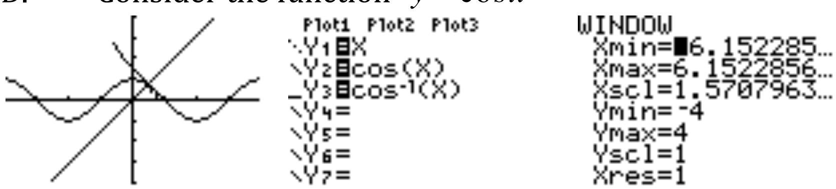


Notice that both the vertical and horizontal tests are passed.

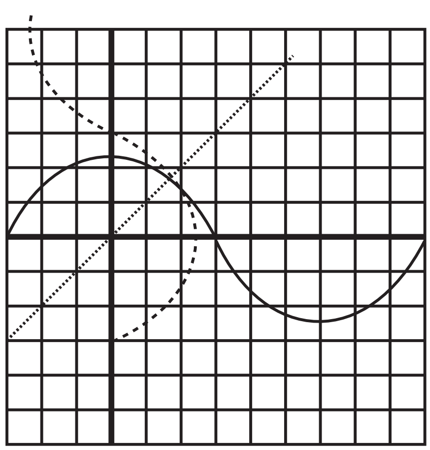
Thus, $y = x$ with the restricted domain of $D_y \; x \geq 0$ is a 1-to-1 function.

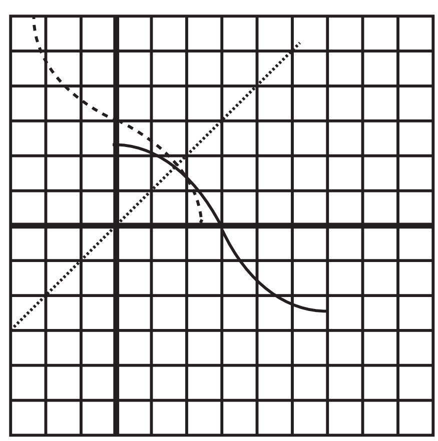
This is particularly important because much of what is done in advanced mathematics is done using 1-to-1 functions.

D. Consider the function $y = \cos x$



however, it is difficult to see the inverse function. In the following view the solid line is cosine and the dotted line is the inverse function. You can see that the inverse is *not* a function. To fix this we restrict the domain of cosine to be $D_{\cos x} \; 0 \leq x \leq \pi$





Notice that the dotted line is a function and that the solid line is a 1-to-1 function.

How did we arrive at these choices for the domain of cosine? If we allowed the domain of cosine to be negative, cosine would fail the horizontal line test. Also, notice that if x is greater than π the graph of cosine heads back up thus failing the horizontal line test. Basically we had no choice for the restricted domain of cosine.

We can think of x as the arc length on a circle therefore when we say $y = \cos^{-1} x$ we can say that y is an arc length; thus the recognition of $\arccos x \Leftrightarrow \cos^{-1} x$.

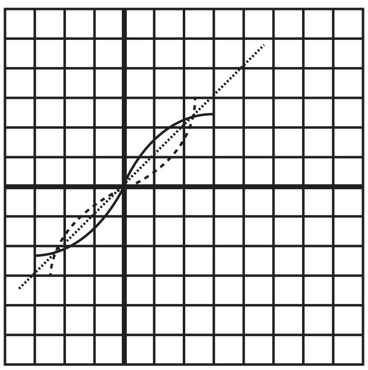
$$\cos^{-1} x = \left\{ (x, y)\, y = \cos^{-1} x \right\}$$

Definition:

domain: $-1 \le x \le 1$

range: $0 \le y \le \pi$

What do you suppose the restriction would have to be with sine?



$D_{\sin^{-1} x}\ -1 \le x \le 1$

$R_{\sin^{-1} x}\ -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

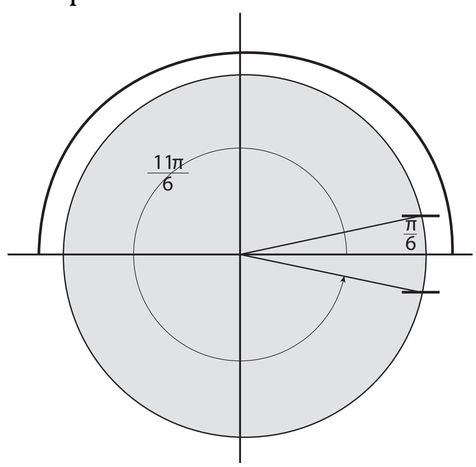
E. Examples:

1. Solve: $\cos^{-1}\frac{\sqrt{3}}{2}=a$

We could "take the cosine of both sides…": $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\cos(a)$

Understanding that the function undoes what the inverse does, that is cosine undoes what inverse cosine does so the left is just $\frac{\sqrt{3}}{2}$ thus we have $\frac{\sqrt{3}}{2}=\cos a$. Now what values of *a* satisfy the equation? $a=\frac{\pi}{6}$ and $a=\frac{11\pi}{6}$. We know that $\frac{\pi}{6}$ is within the range of inverse cosine and that $\frac{11\pi}{6}$ is not in the range of inverse cosine so our answer is $\frac{\pi}{6}$.

This graphic helps us see the possibilities.

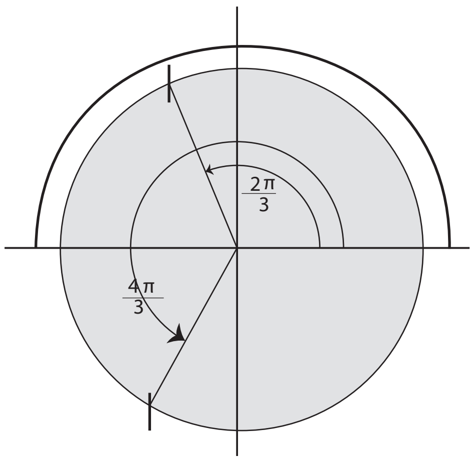


Instead of all of this we could consider $\cos^{-1}\frac{\sqrt{3}}{2}=a$ and the statement: Cosine of what is $\frac{\sqrt{3}}{2}$?

The answer of course is $\frac{\pi}{6}$ and $\frac{11\pi}{6}$ and since we know the range of inverse cosine to be between 0 and π, we would select $\frac{\pi}{6}$.

Example 2: Solve $\cos^{-1}\left(-\frac{1}{2}\right)=a$

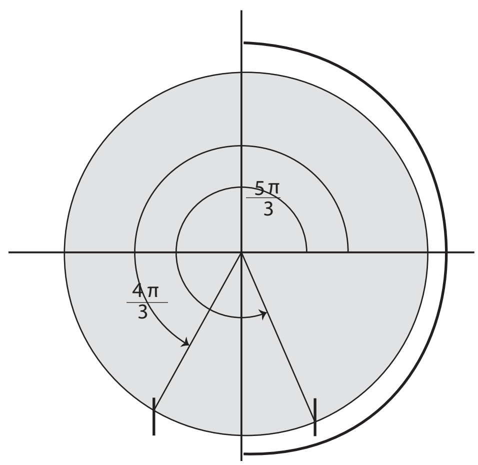
We would take the cosine of both sides and end up with $-\frac{1}{2}=\cos a$ so a could be $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.



Of course we select $a=\frac{2\pi}{3}$.

Example 3: Solve $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=a$

Notice that neither is allowable because the range $R_{\sin^{-1}}\; -\frac{\pi}{2}\le \sin^{-1}x\le\frac{\pi}{2}$ however $-\frac{\pi}{3}$ is in the range and $\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$.



Example 4: Solve $\cos\left(\sin^{-1}\frac{1}{2}\right)=a$

We don't know what $\cos\left(\sin^{-1}\frac{1}{2}\right)$ is but we do know $\sin^2 x+\cos^2 x \;=1$ so $\cos x=\pm\sqrt{1-\sin^2 x}$.

This means that $\cos\left(\sin^{-1}\frac{1}{2}\right)=\pm\sqrt{1-\sin^2\left(\sin^{-1}\frac{1}{2}\right)}$

$$=\pm\sqrt{1-\frac{1}{4}}$$

$$=\pm\sqrt{\frac{3}{4}}$$

$$=\pm\frac{\sqrt{3}}{2}$$

so $\cos\left(\sin^{-1}\frac{1}{2}\right) = \pm\frac{\sqrt{3}}{2}$ but which is it?

Notice that $\sin^{-1}\frac{1}{2}$ is an item fourth or first quadrant (Range of inverse sine: $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$.)
Note that the cosine of anything is either the fourth or first quadrant is positive so the required answer is $\cos\left(\sin^{-1}\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$.

Example 5. Solve $\sin\left(\arccos\frac{3}{5}\right) = a$. As we did in the previous example, $\sin x = \pm\sqrt{1-\cos^2 x}$.
We end with $\sin\left(\arccos\frac{3}{5}\right) = \pm\frac{4}{5}$. And, as above, we notice that inverse cosine is between 0 and 2π – that is in the first or second quadrant and we know that the sine of anything in the first or second quadrant is positive. Thus the correct answer is $\sin\left(\arccos\frac{3}{5}\right) = \frac{4}{5}$.

Example 6. For what values of *u* is this true? $\left(\sqrt{u}\right)^2 = u$

Here the functions are the "squaring function" and the "square root function".

True *only* if $u \ge 0$ because the square root function is not defined for negative values in the domain.

With this basic idea in mind, for what values of *u* are each of the following true?

$a)$ $\cos\left(\cos^{-1} u\right) = u$ $b)$ $\cos^{-1}\left(\cos u\right) = u$

$c)$ $\sin\left(\sin^{-1} u\right) = u$ $d)$ $\sin^{-1}\left(\sin u\right) = u$

Solve for exact values: (no decimal answers)

$e)$ $e = \cos^{-1}\frac{\sqrt{3}}{2}$ $f)$ $f = \sin^{-1}\frac{\sqrt{2}}{2}$ $g)$ $g = \sin^{-1}\frac{1}{2}$

$h)$ $h = \cos^{-1}\left(-\frac{1}{2}\right)$ $i)$ $i = \cos\left(\cos^{-1}\frac{1}{3}\right)$ $j)$ $j = \sin\sin^{-1}\frac{1}{5}$

$k)$ $k = \cos\left(\sin^{-1}\left(-\frac{4}{5}\right)\right)$ $l)$ $l = \sin\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$ $m)$ $m = \sin^{-1}\left(\sin 3\right)$

$n)$ $n = \cos^{-1}\left(\cos 5\right)$